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Structure of space-like submanifolds in pseudo-Euclidean space

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Abstract

Let M be an n-dimensional complete non-compact space-like submanifold in a pseudo-Euclidean Space $\mathbf{R}_p^{n+p}$ of index p. In this paper, we find sufficient conditions for M to have only one end or finitely many ends.

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1. Introduction and main results

Let $\mathbf{R}_p^{n+p}$ be an $(n+p)$-dimensional pseudo-Euclidean space of index n, namely the vector space $\mathbf{R}^{n+p}$ endowed with the metric

$$\langle , \rangle = (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2 - (\mathrm{d}x^{n+1})^2 - \cdots - (\mathrm{d}x^{n+p})^2,$$

where $(x_1, \ldots, x_{n+p})$ are the canonical coordinates in $\mathbf{R}_p^{n+p}$. A smooth immersion $\psi : M \to \mathbf{R}^{n+p}$ of an n-dimensional manifold M is said to be a *space-like submanifold*, if the induced metric via ψ is a Riemannian metric on M which, is also denoted by $\langle , \rangle$.

Space-like submanifolds have been studied extensively. Calabi raised the Bernstein problem for space-like extremal hypersurfaces in Minkowski space $\mathbf{R}_1^{n+1}$, and proved that such

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hypersurfaces have to be hyperplanes when $n \leq 4$ [2]. Cheng and Yau solved the problem for all n [5]. Generalizing an earlier result of Palmer [16], Xin [19] has characterized space-like hyperplanes as the only complete constant mean curvature space-like hypersurfaces in $\mathbf{R}_1^{n+1}$ whose image under the Gauss map is bounded in the hyperbolic space. Later, by an elegant argument the above theorem of Xin was improved by Cao et al. [4] and Xin and Ye [21] independently. They showed that a complete space-like hypersurface of constant mean curvature in $\mathbf{R}_1^{n+1}$ whose image under the Gauss map is contained in a horoball of the hyperbolic space $\mathbf{H}^{n+1}(-1)$ is a hyperplane. By using the maximum principle due to Omori [15], Aledo and Alias [1] have shown that the only complete space-like hypersurfaces of constant mean curvature in $\mathbf{R}_1^{n+1}$ which are bounded between two parallel space-like hyperplanes are the space-like hyperplanes. It should be mentioned that the corresponding result of Aledo–Alias' theorem for minimal surfaces in Euclidean 3-space turns out to be false [8]. On the other hand, Treibergs [18] constructed many non-linear examples of complete space-like hypersurfaces with non-zero constant mean curvature. Recently, in a series of interesting papers, Xin [20] and Jost and Xin [9–11] proved various metric uniqueness theorems for space-like submanifolds in $\mathbf{R}_p^{n+p}$ with parallel mean curvature vectors.

In this paper, we study the topology of space-like submanifolds in $\mathbf{R}_p^{n+p}$ by using a nice idea of Cao et al. in [3]. Our first result can be stated as follows.

Theorem 1.1. *Let M be an $n(\geq 3)$-dimensional complete, non-compact, immersed space-like submanifold in $\mathbf{R}_p^{n+p}$ and denote by $\mathbf{H}$ the mean curvature vector of M. Assume that the Sobolev inequality holds on M, that is, there exists a constant $c > 0$, such that*

$$c\left(\int_M |\psi|^{n/(n-1)}\right)^{(n-1)/n} \leq \int_M |\nabla\psi|, \tag{1.1}$$

for any compactly supported function $\psi \in H_{1,2}(M)$. If the total mean curvature of M satisfies

$$\int_M |\mathbf{H}|^n \leq \left(\frac{(n-2)c}{(n-1)n}\right)^n, \tag{1.2}$$

then M has only one end.

It should be noticed that the submanifolds in Theorem 1.1 are not required to have parallel mean curvature vectors. We believe that the Sobolev inequality actually holds on space-like submanifolds in $\mathbf{R}_p^{n+p}$ with small total mean curvature.

In order to prove Theorem 1.1, we firstly establish a non-existence theorem for non-constant harmonic maps with finite energy. That is, we have

Theorem 1.2. *Let M be an $n(\geq 3)$-dimensional complete, non-compact, immersed space-like submanifold in $\mathbf{R}_p^{n+p}$. Assume that the inequalities (1.1) and (1.2) hold. Then any harmonic map with finite energy from M to a complete manifold with non-positive curvature is a constant. In particular, any harmonic function on M with finite Dirichlet energy is constant.*

Theorem 1.1 is a consequence of Theorem 1.2 and an existence theorem for non-constant harmonic functions with finite Dirichlet energy (see Lemma 3.1).

We fix some notation in order to state our next result. Let M be a complete Riemannian manifold, and let $q : M \to R$ be a differentiable function. Consider the elliptic operator $L = \Delta + q$ associated to the quadratic form:

$$(f, -Lf) = -\int_M fLf = \int_M (|\nabla f|^2 - qf^2).$$

Here $f : M \to R$ is a piecewise smooth function with compact support, Δ is the Laplacian and ∇f is the gradient of f. The index of L is defined to be the supremum, over compact domains of M, of the number of negative eigenvalues of L with Dirichlet boundary condition.

Now we can state our next theorem as follows.

Theorem 1.3. *Let M be an $n(\geq 3)$-dimensional complete, non-compact, immersed space-like submanifold in $\mathbf{R}_p^{n+p}$ and assume that the Sobolev inequality (1.1) holds on M.*

(i) *If the index of the operator $\Delta + (n^2/4)|\mathbf{H}|^2$ is zero, then M has only one end.*
(ii) *If $\Delta + (n^2/4)|\mathbf{H}|^2$ has finite index, then M has finitely many ends.*

2. Preliminaries

In this section, we list some known facts about space-like submanifolds in $\mathbf{R}_p^{n+p}$ and harmonic maps between Riemannian manifolds.

Let M be an n-dimensional complete space-like submanifold in $\mathbf{R}_p^{n+p}$. Choose a local Lorentzian frame field $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}\}$ along M with dual frame field $\{w_1, \dots, w_n, w_{n+1}, \dots, w_{n+p}\}$ such that $e_1, \dots, e_n$, are tangent to M. Let $\{w_{AB}\}_{A,B=1}^{n+p}$ be the corresponding connection forms. We agree with the following range of indices:

$$i, j, k, \dots = 1, \dots, n; \quad s, t, \dots = n+1, \dots, n+p.$$

The induced Riemmanian metric of M is given by $\mathrm{d}s_M^2 = \sum_i w_i^2$ and the structure equations of M are

$$\mathrm{d}w_i = \sum_k w_{ik} \wedge w_k, \quad w_{ij} + w_{ji} = 0,$$

$$\Omega_{ij} = \mathrm{d}w_{ij} - \sum_k w_{ik} \wedge w_{kj} = -\frac{1}{2} \sum_{k,l} R_{ijkl} w_k \wedge w_l,$$

where (R_{ijkl}) is the curvature tensor of M. The Gauss equation of M is given by

$$\mathrm{d}w_{ij} = \sum_k w_{ik} \wedge w_{kj} - \sum_s w_{is} \wedge w_{sj}.$$

By Cartan's lemma we have

$$w_{si} = \sum_j h_{ij}^s w_j,$$

where h_{ij}^s are the components of the second fundamental form of M in $\mathbf{R}_p^{n+p}$. We have from the Gauss equation that

$$R_{ijkl} = -\sum_s (h_{ik}^s h_{jl}^s - h_{il}^s h_{jk}^s).$$

The mean curvature vector of M in $\mathbf{R}_p^{n+p}$ is defined by

$$\mathbf{H} = \frac{1}{n}\sum_{s,i} h_{ii}^s e_s.$$

The Ricci tensor of M is given by

$$R_{ij} = \sum_k R_{kikj} = -\sum_{s,k}(h_{kk}^s h_{jl}^s - h_{kl}^s h_{jk}^s),$$

from which it follows that

$$\mathrm{Ric}_M \geq -\tfrac{1}{4}n^2|\mathbf{H}|^2. \tag{2.1}$$

Now let N be a complete Riemannian manifold of dimension m and let $f : M \to N$ be a harmonic map. Take a local orthonormal frame $\{\bar{e}_\alpha\}_{\alpha=1}^m$ of N and denote by $\{\theta_\alpha\}_{\alpha=1}^m$ the dual coframe and by $\{\theta_{\alpha\beta}\}_{\alpha,\beta=1}^m$ the corresponding connection forms. Let $(K_{\alpha\beta\gamma\delta})$ be the curvature tensor of N; then we have:

$$\mathrm{d}\theta_\alpha = \sum_\beta \theta_{\alpha\beta}\wedge\theta_\beta, \qquad \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0,$$

$$\mathrm{d}\theta_{\alpha\beta} = \sum_\gamma \theta_{\alpha\gamma}\wedge\theta_{\gamma\beta} - \frac{1}{2}\sum_{\gamma,\delta} K_{\alpha\beta\gamma\delta}\theta_\gamma\theta_\delta.$$

Define $f_{\alpha i}$, $1 \leq \alpha \leq m$, $1 \leq i \leq n$ by

$$f^*(\theta_\alpha) = \sum_i f_{\alpha i} w_i. \tag{2.2}$$

The energy density $e(f)$ is given by

$$e(f) = \sum_{\alpha i} f_{\alpha i}^2.$$

Taking the exterior differentiation of (2.2), we get

$$f^*(\mathrm{d}\theta_\alpha) = \sum_i (\mathrm{d}f_{\alpha i}\wedge w_i + f_{\alpha i}\,\mathrm{d}w_i),$$

which gives

$$\sum_i \left(\mathrm{d}f_{\alpha i} - \sum_j f_{\alpha j} w_{ij} - f^*(\theta_{\alpha\beta}) f_{\beta i}\right)\wedge w_i = 0. \tag{2.3}$$

Define $f_{\alpha ij}$ by

$$\mathrm{d} f_{\alpha i} + \sum_{\beta} f_{\beta i} f^*(\theta_{\beta\alpha}) + \sum_{j} f_{\alpha j} w_{ji} = \sum_{j} f_{\alpha ij} w_j. \tag{2.4}$$

Then (2.3) and (2.4) imply that $f_{\alpha ij} = f_{\alpha ji}$ and f is harmonic means

$$\sum_{i} f_{\alpha ii} = 0, \quad \forall\ \alpha = 1, \dots, m.$$

Exterior differentiating (2.4), we get

$$\sum_{l} \left(\mathrm{d} f_{\alpha il} + \sum_{j} (f_{\alpha ij} w_{jl} + f_{\alpha jl} w_{ji}) + \sum_{\beta} f_{\beta il} f^*(\theta_{\beta\alpha}) \right) \wedge w_l$$
$$= \frac{1}{2} \sum_{j,k,l} R_{ijkl} f_{\alpha j} w_k \wedge w_l + \frac{1}{2} \sum_{\beta,\delta,\gamma,k,l} K_{\alpha\beta\gamma\delta} f_{\beta i} f_{\gamma k} f_{\delta l} w_k \wedge w_l. \tag{2.5}$$

Define

$$\sum_{k} f_{\alpha ijk} w_k = \mathrm{d} f_{\alpha ij} + \sum_{k} (f_{\alpha ik} w_{kj} + f_{\alpha kj} w_{ki}) + \sum_{\beta} f_{\beta ij} f^*(\theta_{\alpha\beta});$$

then (2.5) implies that

$$f_{\alpha ikl} - f_{\alpha ilk} = \sum_{j} R_{ijlk} f_{\alpha j} + \sum_{\beta,\gamma,\delta} K_{\alpha\beta\gamma\delta} f_{\beta i} f_{\gamma l} f_{\delta k}.$$

Set $e = e(f)$ and let Δ be the Laplacian operator acting on functions on M. From the above formula, one can easily get the following Bochner type formula for harmonic maps between Riemannian manifolds [6]:

$$\frac{1}{2} \Delta e = \sum_{\alpha,i,j} f_{\alpha ij}^2 + \sum_{\alpha,i,j} R_{ij} f_{\alpha i} f_{\alpha j} - \sum_{\alpha,\beta,\gamma,\delta,i,j} K_{\alpha\beta\gamma\delta} f_{\alpha i} f_{\beta i} f_{\gamma i} f_{\delta j}. \tag{2.6}$$

The following estimate was made by Schoen and Yau in [17]:

$$\sum_{\alpha,i,j} f_{\alpha ij}^2 \geq \left(1 + \frac{1}{2nm} \right) |\nabla \sqrt{e}|^2. \tag{2.7}$$

Recall (see [12]) that an end E of a complete manifold M is non-parabolic if E admits a positive Green's function with Neumann boundary condition.

The following lemma is needed for the proof of Theorem 1.3.

Lemma 2.1. (Li and Tam [13]) *Let M be a complete Riemannian manifold. Let $H_D^0(M)$ be the space of bounded harmonic functions with finite energy and denote by $H^1(L^2(M))$ the first L^2-cohomology of M. Then the number of non-parabolic ends of M is bounded from above by* $\dim H_D^0(M) \leq \dim H^1(L^2(M)) + 1$.

3. Proofs of the results

Proof of Theorem 1.2. Let N be an m-dimensional complete manifold with non-positive sectional curvature and $f : M \to N$ be a harmonic map with finite energy. Denote by e the energy density of f. It follows from (2.1), (2.6) and (2.7) and the non-positivity of the sectional curvature of N that

$$\frac{1}{2}\Delta e \geq \left(1+\frac{1}{2nm}\right)|\nabla\sqrt{e}|^2 - \frac{n^2}{4}|\mathbf{H}|^2 e. \tag{3.1}$$

Let $\psi \in H_{1,2}(M)$ be a compactly supported function. Replacing ψ in (1.1) by $\psi^{2(n-1)/(n-2)}$ and using the Hölder inequality, we arrive at

$$c\left(\int_M |\psi|^{2(n-1)/(n-2)}\right)^{(n-1)/n} \leq \frac{2(n-1)}{n-2}\left(\int_M \psi^{2(n-1)/(n-2)}\right)^{1/2}\left(\int_M |\nabla\psi|^2\right)^{1/2},$$

which gives

$$\left(\int_M |\psi|^{2n/(n-2)}\right)^{(n-2)/n} \leq \frac{4(n-1)^2}{(n-2)^2c^2}\int_M |\nabla\psi|^2 \equiv c'\int_M |\nabla\psi|^2. \tag{3.2}$$

Fix a $p \in M$ and choose ϕ to be a non-negative cut-off function with the properties:

$$\phi = \begin{cases} 1 & \text{on } B(p,r), \\ 0 & \text{on } M \setminus B(p,3r) \end{cases}$$

and

$$|\nabla\phi| \leq \frac{1}{r},$$

where $B(p,r)$ denotes the geodesic ball of radius r with center p. Multiplying (3.1) by ϕ^2 and integrating over M, one gets from the divergence theorem that

$$\begin{aligned}&\left(1+\frac{1}{2nm}\right)\int_M |\nabla\sqrt{e}|^2\phi^2 \\ &\quad\leq \frac{n^2}{4}\int_M \phi^2|\mathbf{H}|^2 e + \frac{1}{2}\int_M \phi^2\Delta e = \frac{n^2}{4}\int_M \phi^2|\mathbf{H}|^2 e - 2\int_M \sqrt{e}\phi\nabla\sqrt{e}\nabla\phi.\end{aligned} \tag{3.3}$$

Set

$$A_0 = \left(\int_M |H|^2\right)^{2/n};$$

then

$$\frac{n^2A_0c'}{4} \leq 1. \tag{3.4}$$

It follows from the Hölder inequality and (3.2) that

$$\int_M \phi^2|\mathbf{H}|^2 e \le \left(\int_M |\mathbf{H}|^n\right)^{2/n}\left(\int_M (\phi\sqrt{e})^{2n/(n-2)}\right)^{(n-2)/n} \le A_0 c' \int_M |\nabla(\phi\sqrt{e})|^2$$
$$= A_0 c' \int_M (|\nabla\phi|^2 e + \phi^2|\nabla\sqrt{e}|^2 + 2\phi\sqrt{e}\nabla\sqrt{e}\nabla\phi). \tag{3.5}$$

Substituting (3.5) into (3.3), one has

$$\left(1 + \frac{1}{2nm} - \frac{n^2 A_0 c'}{4}\right)\int_M \phi^2|\nabla\sqrt{e}|^2$$
$$\le 2\left(\frac{n^2 A_0 c'}{4} - 1\right)\int_M \sqrt{e}\phi\nabla\sqrt{e}\nabla\phi + \frac{n^2 A_0 c'}{4}\int_M |\nabla\phi|^2 e$$
$$\le \left(1 - \frac{n^2 A_0 c'}{4}\right)\left(\int_M e|\nabla\phi|^2 + \int_M \phi^2|\nabla\sqrt{e}|^2\right) + \frac{n^2 A_0 c'}{4}\int_M |\nabla\phi|^2 e. \tag{3.6}$$

Therefore

$$\frac{1}{2nm}\int_M \phi^2|\nabla\sqrt{e}|^2 \le \int_M |\nabla\phi|^2 e,$$

which implies that

$$\int_{B(p,r)} |\nabla\sqrt{e}|^2 \le \int_M \phi^2|\nabla\sqrt{e}|^2 \le 2mn\int_M |\nabla\phi|^2 e \le \frac{2mn}{r^2}\int_{B(p,3r)\setminus B(p,r)} e.$$

Letting $r \to \infty$, the right-hand side tends to 0 since f has finite energy. Hence e is constant. But from the proof of Lemma 3.1 (see below), we know that M has infinite volume. Therefore, we conclude from $E(f) < \infty$ that $e = 0$. This completes the proof of Theorem 1.2. □

It has been shown by Schoen and Yau that any smooth map of finite energy from a complete Riemannian manifold M to a compact manifold with non-positive sectional curvature is homotopic to a harmonic map on each compact set of M. Thus Theorem 1.2 implies immediately the following

Corollary 3.1. *Let M be an $n(\geq 3)$-dimensional complete, non-compact, immersed space-like submanifold in $\mathbf{R}_p^{n+p}$ and let N be a compact manifold with non-positive sectional curvature. Assume that the inequalities (1.1) and (1.2) hold. If $f : M \to N$ is a smooth map with finite energy, then f is homotopic to constant on each compact set.*

As an application of this corollary, one has the following result the proof of which is similar to that of the corollary to Theorem 1 in [17].

Corollary 3.2. *Let M be as in Theorem 1.2 and let D be a compact domain in M with smooth simply connected boundary. Then there exists no non-trivial homomorphism from $\pi_1(D)$ into the fundamental group of a compact manifold with non-positive sectional curvature.*

Theorem 1.1 follows from Theorem 1.2 and the next lemma.

Lemma 3.1. *Let M be an $n(\geq 3)$-dimensional complete non-compact Riemannian manifold. Assume that the Sobolev inequality (1.1) holds on M. If M has at least two ends, then there exists on M a non-constant bounded harmonic function with finite Dirichlet energy.*

Proof. Observe that the inequality (1.1) implies the inequality (3.2). Thus, according to Theorem 2 in [3], it suffices to show that each end of M has infinite volume. Since the Sobolev inequality holds on M, it is known that the isoperimetric inequality holds on M [23]. Thus, there exists a positive constant C_2 such that for any bounded open set $\Omega \subset M$, we have

$$(V(\Omega))^{(n-1)/n} \leq C_2 A(\partial\Omega),$$

where $V(\Omega)$ and $A(\partial\Omega)$ denote the volume of Ω and the area of $\partial\Omega$, respectively. If we let $V(x_0, s) = V(s)$ be the volume of the geodesic ball $B(x_0, s)$ in M, then

$$\frac{\mathrm{d}}{\mathrm{d}s} V(s) = A(\partial B(x_0, s)).$$

Hence, setting $\Omega = B(x_0, s)$ in the isoperimetric inequality, we get

$$V(s)^{(n-1)/n} \leq C_2 V'(s)$$

for all s. Integrating yields $V(s) \geq (nC_2)^{-n} s^n$. Now let $K \subset M$ be a compact subset of M and let E be a non-compact component of $M \setminus K$. If E has finite volume, choose L big enough such that

$$(nC_2)^{-n} L^n > V(E).$$

Let x be a point in E such that $\mathrm{dist}(x, \partial E) \geq L$, then

$$V(E) \geq V(B(x, L)) \geq (nC_2)^{-n} L^n > V(E).$$

This is a contradiction which shows that each end of M has infinite volume. The proof of Lemma 3.1 is completed. □

Proof of Theorem 1.3.

(i) Let N be an m-dimensional complete manifold with non-positive sectional curvature. We shall show that any harmonic map $f : M \to N$ with finite energy is constant. This fact, combining with Lemma 3.1, will imply that M has only one end. Since the index of $\Delta + (n^2/4)|\mathbf{H}|^2$ is zero, we know from the definition that

$$\int_M |\nabla\psi|^2 \geq \frac{n^2}{4} \int_M |\mathbf{H}|^2 \psi^2$$

for any compactly supported $\psi \in H_{1,2}(M)$.

Let $e = e(f)$. Replacing ψ by $\sqrt{e}$ with $\phi \in C_0^\infty$ we obtain

$$\begin{aligned}\frac{n^2}{4}\int_M |\mathbf{H}|^2 e\phi^2 &\le \int_M e|\nabla\phi|^2 + \int_M \phi^2|\nabla\sqrt{e}|^2 + 2\int_M \sqrt{e}\phi\nabla\sqrt{e}\phi \\ &= \int_M e|\nabla\phi|^2 + \int_M \phi^2|\nabla\sqrt{e}|^2 - \frac{1}{2}\int_M \phi^2\Delta e. \end{aligned}\tag{3.7}$$

Observe that (3.1) also holds on our M. Thus, we have (cf. (3.3)):

$$-\frac{1}{2}\int_M \phi^2\Delta e \le \frac{n^2}{4}\int_M |\mathbf{H}|^2 e\phi^2 - \left(1 + \frac{1}{2nm}\right)\int_M |\nabla\sqrt{e}|^2\phi^2. \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\frac{1}{2nm}\int_M |\nabla\sqrt{e}|^2\phi^2 \le \int_M e|\nabla\phi|^2.$$

Choosing the same function ϕ as in Theorem 1.2 we obtain

$$\int_{B(p,r)} |\nabla\sqrt{e}|^2 \le \frac{2nm}{r^2}E(f).$$

Letting $r \to \infty$ we find that e is a constant. Since the Sobolev inequality (1.1) holds on M, we know from the proof of Lemma 3.1 that $\mathrm{vol}(M) = \infty$. Thus $e = 0$. Observe that if we do not assume that the Sobolev inequality (1.1) holds on M, we can still conclude that $e = 0$. In fact, one obtains by substituting the above ψ into (3.7) and using the fact that e is constant that

$$\frac{n^2}{4}\int_{B(p,r)} |\mathbf{H}|^2 e \le \frac{1}{r^2}\int_{B(p,3r)\setminus B(p,r)} e.$$

If $e \ne 0$, then we get by letting $r \to \infty$ that $\mathbf{H} \equiv 0$. Hence, (2.1) implies that M has non-negative Ricci curvature and so $\mathrm{vol}(M) = \infty$ [22]. This is a contradiction since $E(f) < \infty$. Hence f is a constant.

(ii) Since $\Delta + (n^2/4)|\mathbf{H}|^2$ has finite index, one can use the same arguments as in [7] to show that there exists a compact set $\Omega \subset M$ such that the operator $\Delta + (n^2/4)|\mathbf{H}|^2$ when restricted to compactly supported $H_{1,2}$ functions on $M \setminus \Omega$ has index zero. This is equivalent to say

$$\int_{M\setminus\Omega} |\nabla\psi|^2 \ge \frac{n^2}{4}\int_{M\setminus\Omega} |\mathbf{H}|^2\psi^2 \tag{3.9}$$

for all compactly supported $H_{1,2}$ function ψ on $M \setminus \Omega$. We can assume that $\Omega \subset B(p, R_0)$ for some $p \in M$ and $R_0 > 0$. The monotonicity of eigenvalues [7] implies that for any $\psi \in H_{1,2}(M \setminus B(p, R_0))$ with compact support, it holds

$$\int_{M\setminus B(p,R_0)} |\nabla\phi|^2 \ge \frac{n^2}{4}\int_{M\setminus B(p,R_0)} |\mathbf{H}|^2\phi^2. \tag{3.10}$$

By choosing $\psi = \phi h$ with ϕ being a non-negative compactly supported function on $M \setminus B(p, R_0)$, we have

$$\begin{aligned} &\frac{n^2}{4}\int_{M\setminus B(p,R_0)} \phi^2|\mathbf{H}|^2h^2 \\ &\quad \le \int_{M\setminus B(p,R_0)} |\nabla\phi|^2h^2 + 2\int_{M\setminus B(p,R_0)} \phi h\nabla\phi\nabla h + \int_{M\setminus B(p,R_0)} \phi^2|\nabla h|^2 \\ &\quad = \int_{M\setminus B(p,R_0)} |\nabla\phi|^2h^2 - \int_{M\setminus B(p,R_0)} \phi^2 h\Delta h. \end{aligned} \tag{3.11}$$

In the proof of Lemma 3.1, we showed that each end of M has infinite volume. Since the Sobolev inequality (1.1) holds on M, we can use the same arguments as in the proof of Theorem 3 in [14] to show that each end of M is non-parabolic. Thus according to Lemma 2.1, in order to show that M has finitely many ends, we need only to show that M has finite first L^2-Betti number, i.e. $\dim H^1(L^2(M)) < \infty$. For any L^2 harmonic 1-form w on M, let $h = |w|$ be the length of w and denote by w^* be the vector field dual to w. It follows from the Bochner formula and (2.1) that

$$\frac{1}{2}\Delta h^2 = \mathrm{Ric}(w^*, w^*) + |\nabla w|^2 \ge -\frac{n^2}{4}|\mathbf{H}|^2h^2 + |\nabla w|^2,$$

where ∇w denotes the covariant derivative of w.

By using the same arguments as in the proof of Theorem 5 in [14], we have

$$|\nabla w|^2 \ge \frac{n|\nabla h|^2}{n-1}.$$

Hence,

$$h\Delta h \ge -\frac{n^2}{4}|\mathbf{H}|^2h^2 + \frac{|\nabla h|^2}{n-1}. \tag{3.12}$$

Substituting (3.12) into (3.11), we get

$$\int_{M\setminus B(p,R_0)} \phi^2|\nabla h|^2 \le (n-1)\int_{M\setminus B(p,R_0)} |\nabla\phi|^2h^2. \tag{3.13}$$

Since (1.1), (3.12) and (3.13) hold on M, one can then use the same discussions as in the proof of Theorem 5 in [14] to show that $\dim H^1(L^2(M)) < \infty$. This completes the proof of Theorem 1.3. □

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